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On Explicit Formulas of Hyperbolic Matrix Functions

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Abstract

Hyperbolic matrix functions are essential for solving hyperbolic coupled partial differential equations. In fact the best analytic-numerical approximations for resolving these equations come from the use of hyperbolic matrix functions. The hyperbolic matrix sine and cosine $sh(A)$, $ch(A)$ $(A \in M_r(\mathbb{C}))$ can be calculated using numerous different techniques. In this article we derive some explicit formulas of $sh(tA)$ and $ch(tA)$ $(t \in \mathbb{R})$ using the Fibonacci-Hörner and the polynomial decomposition, these decompositions are calculated using the generalized Fibonacci sequences combinatorial properties in the algebra of square matrices. Finally we introduce a third approach based on the homogeneous linear differential equations. And we provide some examples to illustrate your methods.

Keywords: matrix functions; generalized Fibonacci sequence; hyperbolic matrix functions; Fibonacci-Horner decomposition.

1 Introduction

Numerous fields of mathematics, engineering and applied science use the hyperbolic matrix functions. Notably these functions are used as the solution of coupled partial differential systems of hyperbolic type, [3, 1].

The hyperbolic matrix functions can be defined for every matrix $A \in M_r(\mathbb{C})$ by :

$$ch(A) = \sum_{n\geq 0} \frac{A^{2n}}{(2n)!}, \qquad sh(A) = \sum_{n\geq 0} \frac{A^{2n+1}}{(2n+1)!}. \tag{1}$$

Hyperbolic sine and hyperbolic cosine represent the odd part and the even part of the exponential function respectively, that is

$$ch(A) = \frac{e^A + e^{-A}}{2}, \qquad sh(A) = \frac{e^A - e^{-A}}{2}. \tag{2}$$

For computing hyperbolic matrix functions, a wide range of approaches and techniques have been developed to provide various effective expressions, notably methods and algorithms based on Hermite matrix polynomials have been extensively studied [6, 8]. Another approach utilizing orthogonal matrix polynomials has been introduced in [7]. And since the Taylor series expansion of the hyperbolic matrix functions have an infinite radius of convergence a Schur-Parlett algorithm can be employed which uses a Schur decomposition with reordering and blocking followed by the block form of a recurrence of Parlett to approximate the values of the hyperbolic matrix functions, [5].

Coupled partial differential problems often appear in a broad range of technical and scientific fields.

Using a series that utilized hyperbolic matrix functions, we could construct an exact solution of coupled hyperbolic systems of the type:

$$\begin{cases} v_{tt}(x,t) - M_1 v_{xx}(x,t) = 0, & 0 < x < 1, \quad t > 0, \\ v(0,t) + N_1 v_x(0,t) = 0, & t > 0, \\ M_2 v(1,t) + N_2 v_x(1,t) = 0, & t > 0, \\ v(x,0) = g(x), & 0 \leq x \leq 1, \\ v_t(x,0) = h(x), & 0 \leq x \leq 1, \end{cases}$$

where M_1, N_1, M_2 and N_2 are in $M_r(\mathbb{C})$, and the unknown v, g and h are r-vector valued functions (see [9]).

The computation of hyperbolic matrix functions is still a fascinating subject. The primary goal of this paper is to offer three basic techniques for the computation of $ch(tA)$ and $sh(tA)$ for every $A \in GL_r(\mathbb{C})$ the first method uses the Fibonacci-Hörner decomposition of the matrix powers (see [11, 2, 4]), to calculate some specific formulas of $ch(tA)$ and $sh(tA)$ for every $A \in GL_r(\mathbb{C})$. The second method foundational tools are the dynamic solution of the r-th order scalar differential equations and the combinatorial formulation of r-generalized Fibonacci sequences.[15]. The last method uses the homogeneous linear differential equations.

Fibonacci-Horner's algorithm for evaluating a random polynomial is the most efficient one. Alexander Ostrowski and Victor Pan demonstrated how to calculate a degree n polynomial using

only $O(\sqrt{n})$ and $O(n)$ scalar multiplications, however this method is not the most efficient when applied to a matrix. In fact a matrix polynomial of degree n with square matrix $A \in M_m(\mathbb{C})$ as variables can be evaluated in $O(m^{2.3}\sqrt{n} + m^2 n)$ time. If $m = n$, this is less than $O(n^3)$.

2 Decomposition of the Hyperbolic Matrix Functions via Recursive Relations

2.1 Linear recurrence relations in the space of square matrix

Let $A \in M_r(\mathbb{C})$ with characteristic polynomial $P(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-1}$. By using the Cayley-Hamilton theorem we obtain $P(A) = \Theta_r$ (the null matrix of order $r \times r$). Consequently, the matrix sequence $\{A^n\}_{n\geq 0}$ is a r-generalized Fibonacci sequence in $M_r(\mathbb{C})$, its initial conditions and coefficients are $A^0 = I_r$ (the matrix identity), $A, \cdots, A^{r-1}$ and $a_0, \cdots, a_{r-1}$, respectively. When A is nonsingular, we consider the r- generalized Fibonacci sequence $\{A^n\}_{n\geq -r+1}$, out of practicality, its initial conditions and coefficients are $A^{-r+1}, \cdots, A^{-1}$, $A^0 = I_r$ and $a_0, \cdots, a_{r-1}$, respectively. following the findings of [15], we get

$$A^n = u_A(n)A_0 + u_A(n-1)A_1 + \cdots + u_A(n-r+1)A_{r-1}, \qquad \text{for every } \ n \geq r, \tag{3}$$

with

$$A_0 = I_r, \qquad A_i = A^i - a_0 A^{i-1} - \cdots - a_{i-1} I_r, \qquad \text{for every } i = 1, \cdots, r-1. \tag{4}$$

We define the sequence $\{u_A(n)\}_{n\geq -r+1}$ by,

$$u_A(n) = \sum_{k_0+2k_1+\cdots+rk_{r-1}=n} \frac{(k_0+k_1+\cdots+k_{r-1})!}{k_0!k_1!\cdots k_{r-1}!} a_0^{k_0} a_1^{k_1} \cdots a_{r-1}^{k_{r-1}}, \tag{5}$$

for all $n \geq 1$, with $u_A(0) = 1$ and $u_A(n) = 0$ for $-r+1 \leq n \leq -1$. In [15, 13] It was established that the sequence $\{u_A(n)\}_{n\geq -r+1}$ satisfies the linear recurrence relation given by :

$$u_A(n+1) = a_0 u_A(n) + a_1 u_A(n-1) + \cdots + a_{r-1} u_A(n-r+1), \text{ for every } n \geq 0. \tag{6}$$

Simply put, $\{u_A(n)\}_{n\geq 1-r}$ is a r-generalized Fibonacci sequence.

Actually, a straightforward proof by induction demonstrates that the decomposition (3) of the powers A^n $(n \geq r)$ is still valid for every $A \in M_r(\mathbb{C})$ (singular or nonsingular), with the A_k $(0 \leq k \leq r-1)$ given by (4) and the sequence $\{u_A(n)\}_{n\geq 1-r}$ by (5).

The set of matrices $\{A_0\ ,\ A_1\ , \cdots\ ,\ A_{r-1}\}$ is known as the *Fibonacci-Horner basis* of the power decomposition (3) of A.

The coefficients of $u_A(n)(n \geqslant 1)$ are obtained using the recursive formula (6). However, we can also calculate them using the determinant of the $n \times n$ upper Hessenberg matrix [10, 17], given by

$$u_A(n) = \begin{vmatrix} a_0 & a_1 & \dots & a_{r-1} & 0 & \dots & 0 \\ -1 & a_0 & \dots & a_{r-2} & a_{r-1} & \dots & 0 \\ \vdots & \vdots & \dots & \dots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \ddots & \dots & \dots & \vdots \\ 0 & 0 & \dots & \dots & \dots & a_0 & a_1 \\ 0 & 0 & \dots & \dots & \dots & -1 & a_0 \end{vmatrix}. \tag{7}$$

For $r+1 \leqslant n$, expression (7) is the consequence of the determinant of a Toeplitz ($r+1$) banded matrix.

A determining representation for the n-th power matrix $A^n (n \geqslant r)$ in the power basis is given by

$$A^n = \sum_{k=0}^{r-1} u_{n-r+1}^{(k)} A^{r-1-k} = \sum_{k=0}^{r-1} u_{n-r+1}^{(r-1-k)} A^k, \tag{8}$$

where $u_A^{(k)}(n-r+1)$, is the determinant of the matrix given by the expression (7), with the first row shifted k positions forward, (see Proposition 2.2 [14]).

2.2 Fibonacci-Hörner decomposition of the hyperbolic matrix functions

In the flowing theorem we will introduce the Fibonacci-Hörner decomposition of the hyperbolic sine and cosine matrix functions.

Theorem 2.1. *Let $A \in M_r(\mathbb{C})$ and $R_A(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-1}$ an annihilator polynomial of A. Let $\{A_i\}_{0 \leq i \leq r-1}$ be the Fibonacci-Hörner basis of A. Then, we have*

$$ch(tA) = \sum_{i=0}^{r-1} \phi_i^c(t) A_i, \quad \text{with} \quad \phi_i^c(t) = \sum_{n \geq 0} \frac{t^{2n}}{(2n)!} u_A(2n-i), \tag{9}$$

$$sh(tA) = \sum_{i=0}^{r-1} \phi_i^s(t) A_i, \quad \text{with} \quad \phi_i^s(t) = \sum_{n \geq 0} \frac{t^{2n+1}}{(2n+1)!} u_A(2n+1-i), \tag{10}$$

with $u_A(n)$ is given by (5).

Proof. By substituting (3) in the series expansion of the hyperbolic cosine function we obtain

$$\begin{aligned} ch(tA) &= \sum_{n \geq 0} \frac{t^{2n}}{(2n)!} A^{2n}, \\ &= \sum_{n \geq 0} \frac{t^{2n}}{(2n)!} \sum_{i=0}^{r-1} u_A(2n-i) A_i, \\ &= \sum_{i=0}^{r-1} [\sum_{n \geq 0} \frac{t^{2n}}{(2n)!} u_A(2n-i)] A_i, \\ &= \sum_{i=0}^{r-1} \phi_i^c(t) A_i. \end{aligned}$$

Applying the same approach we obtain the Fibonacci-Hörner decomposition hyperbolic sinus function. □

Example 2.1. (*The boundary problem*)
In this example we will calculate a solution of the he boundary value problem

$$\begin{cases} v_{tt}(x,t) - M_1 v_{xx}(x,t) = 0, & 0 < x < 1, \quad t > 0, \\ v(0,t) + N_1 v_x(0,t) = 0, & t > 0, \\ M_2 v(1,t) + N_2 v_x(1,t) = 0, & t > 0, \end{cases}$$

of a particular form

$$v(x,t) = A(t)B(x), \qquad A(t) \in M_2(\mathbb{C}), \qquad B(x) \in \mathbb{C}^2.$$

With $M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{R})$,

$R\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{R})$ *is a square root of the matrix* $-M_1$ ($R^2 = -M_1$).

In [9] *it was shown that B and A satisfy :*

$$A''(t) + \alpha^2 M_1 A(t) = 0, \quad \alpha \geqslant 0, \quad t \geqslant 0, \tag{11}$$
$$B''(x) + \alpha^2 B(x) = 0, \quad \alpha \geqslant 0, \quad 0 < x < 1. \tag{12}$$

A general $M_2(\mathbb{C})$ *solution of* (11) *is given by*

$$A(t,\lambda) = X_1(t,\alpha)D(\alpha) + X_2(t,\alpha)E(\alpha), \qquad D(\alpha), E(\alpha) \in M_2(C).$$

With

$$X_1(t,\alpha) = \begin{cases} ch(\alpha R t), & \alpha > 0, \\ I, & \alpha = 0, \end{cases}, \qquad X_2(t,\alpha) = \begin{cases} (\alpha R)^{-1} sh(\alpha R t), & \alpha > 0, \\ tI, & \alpha = 0. \end{cases}$$

Using the Fibonacci-Hörner decomposition of the hyperbolic matrix functions we get,

$P(z) = z^2 + 1$ *is an annihilator polynomial of R. Thus we have* $a_0 = 0$, $a_1 = -1$ *and the Fibonacci-Hörner basis is* $R_0 = I_2$ *and* $R_1 = R$.

The sequence $(u_R(n))_{n \geq -1}$ *is given by* $\begin{cases} u_R(l) = (-1)^{\frac{l}{2}}, & \text{if } l \text{ is even}, \\ u_R(l) = 0, & \text{if } l \text{ is odd}. \end{cases}$

A direct computation using the expression (9 *and* 10) *yields:*

$$ch(\alpha R t) = \phi_0^c(\alpha t)R_0 + \phi_1^c(\alpha t)R_1 = cos(\alpha t)I_2,$$
$$sh(\alpha R t) = \phi_0^c(\alpha t)R_0 + \phi_1^c(\alpha t)R_1 = sin(-\alpha t)R.$$

And a general $\mathbb{C}^2$ *solution of* (12) *is given by*

$$B(x,\alpha) = \begin{cases} \sin(\alpha x)D'(\alpha) + \cos(\alpha x)E'(\alpha), & \alpha > 0, \\ xD'(0) + E'(0), & \alpha = 0. \end{cases}$$

2.3 Polynomial decomposition of the hyperbolic matrix functions

In this section we will introduce two approaches for calculating the polynomial decomposition of the hyperbolic matrix functions, the first approach is based in the representation for the n-th power matrix (8) the second method is dependent on the the polynomial decomposition of the exponential matrix function already established in [16].

Theorem 2.2. *Let* $A \in M_d(\mathbb{C})$, *and* $R(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-1} (2 \leqslant r \leqslant d)$ *an annihilator polynomial of A. Then, for all* $t \in \mathbb{R}$, *we have,*

$$ch(tA) = \sum_{k=0}^{r-1} \Phi_k(t) A^k, \tag{13}$$

and

$$sh(tA) = \sum_{k=0}^{r-1} \Psi_k(t) A^k. \tag{14}$$

where

$$\Phi_k(t) = \begin{cases} \frac{t^k}{k!} + \sum_{n=[\frac{r+1}{2}]}^{+\infty} u_{2n-r+1}^{(r-1-k)} \frac{t^{2n}}{(2n)!}, & \text{if } k \text{ is even,} \\ \sum_{n=[\frac{r+1}{2}]}^{+\infty} u_{2n-r+1}^{(r-1-k)} \frac{t^{2n}}{(2n)!}, & \text{if } k \text{ is odd.} \end{cases},$$

and

$$\Psi_k(t) = \begin{cases} \sum_{n=[\frac{r}{2}]}^{+\infty} u_{2n-r+2}^{(r-1-k)} \frac{t^{2n+1}}{(2n+1)!}, & \text{if } k \text{ is even,} \\ \frac{t^k}{k!} + \sum_{n=[\frac{r}{2}]}^{+\infty} u_{2n-r+2}^{(r-1-k)} \frac{t^{2n+1}}{(2n+1)!}, & \text{if } k \text{ is odd.} \end{cases}$$

and the coefficients $u_A^{(r-1-s)}(n-r+1)$ *are as given in* (8).

Proof. From (1) we obtain, $ch(tA) = \sum_{n=0}^{[\frac{r-1}{2}]} \frac{t^{2n}}{(2n)!} A^{2n} + \sum_{n=[\frac{r+1}{2}]}^{+\infty} \frac{t^{2n}}{(2n)!} A^{2n}$, and

$sh(tA) = \sum_{n=0}^{[\frac{r-2}{2}]} \frac{t^{2n+1}}{(2n+1)!} A^{2n+1} + \sum_{n=[\frac{r}{2}]}^{+\infty} \frac{t^{2n+1}}{(2n+1)!} A^{2n+1}$.

Therefore, by using (8) we deduce that

$ch(tA) = \sum_{k=0}^{[\frac{r-1}{2}]} \frac{t^{2k}}{(2k)!} A^{2k} + \sum_{n=[\frac{r+1}{2}]}^{+\infty} \frac{t^{2n}}{(2n)!} \sum_{k=0}^{r-1} u_{2n-r+1}^{(r-1-k)} A^k$, and

$sh(tA) = \sum_{n=0}^{[\frac{r-2}{2}]} \frac{t^{2n+1}}{(2n+1)!} A^{2n+1} + \sum_{n=[\frac{r}{2}]}^{+\infty} \frac{t^{2n+1}}{(2n+1)!} \sum_{k=0}^{r-1} u_{2n-r+2}^{(r-1-k)} A^k$.

Considering the uniform convergence, we get

$$ch(tA) = \sum_{k=0}^{r-1} \Phi_k(t) A^k, \quad \text{and} \quad sh(tA) = \sum_{k=0}^{r-1} \Psi_k(t) A^k.$$

□

Another approach for computing the polynomial decomposition of the hyperbolic matrix functions is using the polynomial decomposition of the exponential matrix function.

Let $A \in M_r(\mathbb{C})$ whose characteristic polynomial is given by $P_A(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-1}$, with $a_{r-1} \neq 0$. and $t \in \mathbb{R}$, the Fibonacci-Hörner decomposition of the Matrix Exponential is given by

$$e^{tA} = \sum_{k=0}^{r-1} A^k \left[\frac{t^k}{k!} + \sum_{j=0}^{k} a_{r-1+j-k} \rho_A^{(r-1-j)}(t) \right], \tag{15}$$

where $\rho^{(k)}$ refers to the derivative of order k of the function

$$\rho_A(t) = \sum_{n=0}^{+\infty} \frac{t^{n+r-1}}{(n+r-1)!} u_A(n), \tag{16}$$

(see [16] for proof).

Theorem 2.3. *Let $A \in M_r(\mathbb{C})$, and $t \in \mathbb{R}$ and $P_A(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-1}$, with $a_{r-1} \neq 0$ be its characteristic polynomial. We have :*

$$sh(tA) = \sum_{i=0}^{[\frac{r-2}{2}]} \frac{t^{2i+1}A^{2i+1}}{(2i+1)!} + \sum_{i=0}^{r-1} A^i \sum_{j=0}^{k} a_{r-1+j-i}\psi_s^{(r-i-j)}(t), \tag{17}$$

and

$$ch(tA) = \sum_{i=0}^{[\frac{r-1}{2}]} \frac{t^{2i}A^{2i}}{(2i)!} + \sum_{i=0}^{r-1} A^i \sum_{j=0}^{i} a_{r-1+j-i}\psi_c^{(r-i-j)}(t). \tag{18}$$

Where $\psi_s^{(k)}$ and $\psi_c^{(k)}$ refers to the derivative of order k of the functions defined by :

$$\psi_s(t) = \sum_{n=0}^{+\infty} \frac{t^{2(n+r)-1}}{(2(n+r)-1)!} u_A(2n), \tag{19}$$

$$\psi_c(t) = \sum_{n=0}^{+\infty} \frac{t^{2(n+r)}}{(2(n+r))!} u_A(2n+1). \tag{20}$$

Proof. Using the formula $ch(tA) = \dfrac{e^{tA} + e^{-tA}}{2}$ and $sh(tA) = \dfrac{e^{tA} - e^{-tA}}{2}$.

Applying (15) to the matrix A and $-A$ we obtain

$$ch(tA) = \frac{1}{2}\sum_{k=0}^{r-1} A^k \left[\frac{t^k + (-t)^k}{k!} + \sum_{j=0}^{k} a_{r-1+j-k}(\rho_A^{(r-1-j)}(t) + \rho_A^{(r-1-j)}(-t))\right],$$

$$sh(tA) = \frac{1}{2}\sum_{k=0}^{r-1} A^k \left[\frac{t^k - (-t)^k}{k!} + \sum_{j=0}^{k} a_{r-1+j-k}(\rho_A^{(r-1-j)}(t) - \rho_A^{(r-1-j)}(-t))\right].$$

A direct computation yields the desired results. □

2.4 Hyperbolic matrix functions and homogeneous linear differential equations

In this section we will introduce an additional method for calculating the hyperbolic matrix functions that uses the solutions of homogeneous linear differential equations.

Theorem 2.4. *Let $A \in M_r(\mathbb{C})$, we have*

$$e^{tA} = \sum_{k=0}^{r-1} x_k(t) A^k, \tag{21}$$

with $x_k,\ k \in \{0, 1, 2, ..., r-1\}$ are the solution to the ordinary differential equation :

$$y^{(r)}(t) - a_0 y^{(r-1)}(t) - a_1 y^{(r-2)}(t) - ... - a_{r-1} y(t) = 0.$$

With the initial conditions : $y_k^{(l)}(0) = \delta_{kl}$, (The Kronecker symbol) for all $l \in \{0, 1, 2, ..., r-1\}$

(For proof and an examples, see [12]).

We provide the following examples to illustrate our approach:

Example 2.2. *Let* $A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix} \in M_2(\mathbb{R})$.
Its characteristic polynomial is given by $P_A(x) = x^2 - x - 2$.

We consider the differential equation $y''(x) - y'(x) - 2y(x) = 0$, *its solutions are the functions* $y : t \mapsto \alpha_1 e^{-t} + \alpha_2 e^{2t}$, *(with* α_1, α_2 *are constant).*

A solution of the differential equation verifying the conditions $x_0(0) = 1, x_0'(0) = 0$ *is:*

$$x_0 : t \mapsto \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}.$$

A solution of the differential equation verifying the conditions $x_1(0) = 0, x_1'(0) = 1$ *is:*

$$x_1 : t \mapsto -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t}).$$

By applying the formula (21), we get:

$$e^{tA} = (2e^{it} - e^{2it})I_2 + (e^{2it} - e^{it})A,$$

and

$$e^{-tA} = (2e^{-it} - e^{-2it})I_2 + (e^{-2it} - e^{-it})A.$$

At the end, we have the expressions for $ch(tA)$ *and* $sh(tA)$

$$ch(tA) = \left[\frac{2}{3}ch(t) + \frac{1}{3}ch(2t)\right]I + \left[-\frac{1}{3}ch(t) + \frac{1}{3}ch(2t)\right]A,$$

and

$$sh(tA) = \left[\frac{2}{3}sh(t) + \frac{1}{3}sh(2t)\right]I + \left[-\frac{1}{3}ch(t) + \frac{1}{3}ch(2t)\right]A.$$

Example 2.3. *Let* $A = \begin{pmatrix} 37 & 15 & -19 & 4 \\ -11 & -3 & 4 & 1 \\ 63 & 27 & -34 & 9 \\ 19 & 9 & -11 & 4 \end{pmatrix} \in M_4(\mathbb{R})$, *applying the previous approach, we obtain :*
$sh(tA) = z_0(t)I_4 + z_1(t)A + z_2(t)A^2 + z_3(t)A^3$,

$$\text{with} \begin{cases} z_0(t) = 1, \\ z_1(t) = -\frac{5}{6} + \frac{1}{2}sh(-t) + \frac{1}{2}sh(2t) - \frac{1}{6}sh(3t), \\ z_2(t) = \frac{5}{6} + \frac{5}{12}sh(-t) + \frac{1}{3}sh(2t) - \frac{1}{12}sh(3t), \\ z_3(t) = \frac{1}{6} - \frac{1}{12}sh(-t) - \frac{1}{6}sh(2t) + \frac{1}{12}sh(3t). \end{cases}$$

3 Conclusion

The definition of matrix function using interpolating polynomials has been widely explored in the literature. In the study of matrix functions, the theory of constituent matrices is crucial, and it continues to be a topic of research.

This paper looked at three different ways to calculate the hyperbolic trigonometric matrix. the first method uses the Fibonacci-Hörner decomposition, the second uses the polynomial decomposition of the hyperbolic matrix functions, an additional method is given that uses the homogeneous linear differential equations. In Section 3, we provide two examples from the literature to demonstrate our findings. It's worth noting that our findings can be applied to various types of common functions, such as the resolvent of a matrix.

When we compare the formulas established in the previous sections to the literature on the subject, we believe they are novel. As far as we are aware, our approach differs from those being examined. it is worth exploring the idea of creating an algorithm based in our research, and exploring the applicability of these methods to other matrix functions

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