Malaysian Journal of Mathematical Sciences 16(1): 25–36 (2022)
https://doi.org/10.47836/mjms.16.1.03

Malaysian Journal of Mathematical Sciences

Journal homepage: https://einspem.upm.edu.my/journal

Edge Irregular Reflexive Labeling for Some Classes of Plane Graphs

Yoong, K. K.[1], Hasni, R.[*1], Lau, G. C.[2], and Irfan, M.[3]

[1] *Special Interest Group on Modelling and Data Analytics, Faculty of Ocean Engineering Technology and Informatics, Universiti Malaysia Terengganu, Malaysia*
[2] *Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA (Segamat Campus), Malaysia*
[3] *Department of Mathematics, University of Okara, Pakistan*

E-mail: hroslan@umt.edu.my
[*] *Corresponding author*

Received: 17 June 2021
Accepted: 14 October 2021

Abstract

For a graph G, we define a total k-labeling φ as a combination of an edge labeling $\varphi_e(x) \rightarrow \{1, 2, \ldots, k_e\}$ and a vertex labeling $\varphi_v(x) \rightarrow \{0, 2, \ldots, 2k_v\}$, such that $\varphi(x) = \varphi_v(x)$ if $x \in V(G)$ and $\varphi(x) = \varphi_e(x)$ if $x \in E(G)$, where $k = \max\{k_e, 2k_v\}$. The total k-labeling φ is called an *edge irregular reflexive k-labeling* of G, if for every two edges xy, $x'y'$ of G, one has $wt(xy) \neq wt(x'y')$, where $wt(xy) = \varphi_v(x) + \varphi_e(xy) + \varphi_v(y)$. The smallest value of k for which such labeling exists is called a *reflexive edge strength* of G. In this paper, we study the edge irregular reflexive labeling on plane graphs and determine its reflexive edge strength.

Keywords: edge irregular reflexive labeling; plane graphs; reflexive edge strength.

1 Introduction

All graphs considered in this paper are simple, finite, and undirected with a vertex set $V(G)$ and an edge set $E(G)$. It is all known that a simple graph is impossible to completely irregular, which is to have all vertices of distinct degrees. However, it is possible in multigraphs. Therefore, Chartrand et al. [3] proposed an *irregular assignment* by replacing the number of edges incident on every vertex of a multigraph to the edge labels of a simple graph. They defined the irregular assignment by labeling a set of positive integers $\{1, 2, \ldots, k\}$ to the edges of a graph G of order at least three, such that every vertex weight is distinct, where the vertex weight is a sum of the labels of edges that incident to its vertex. The minimum k for which the graph G has the irregular assignment is called an *irregularity strength* of the graph G, denoted by $s(G)$.

Bača et al. [2] defined a total k-labeling $\rho : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ to be an *edge irregular total k-labeling* of a graph G, if for every two edges xy and $x'y'$ of the graph G, one has $wt(xy) \neq wt(x'y')$, where $wt(xy) = \rho(x) + \rho(xy) + \rho(y)$. The *total edge irregularity strength* of the graph G, denoted by $tes(G)$ is defined as the minimum k for which the graph G has an edge irregular total k-labeling. For a comprehensive survey of graph labelings, please refer [4].

Inspired by the problems of the natural irregular multigraph [3] and the edge irregular total labeling [2], Tanna et al. [8] subsequently combined these problems by allowing for the vertex labels representing vertex degrees contributed by the loops. They noticed that (a) the vertex labels are non-negative even integers, which also representing the fact that each loop contributes 2 to the vertex degree; and (b) vertex label 0 is permissible as representing a loopless vertex.

Hence, Tanna et al. [8] defined a total k-labeling φ as a combination of an edge labeling $\varphi_e : E(G) \to \{1, 2, \ldots, k_e\}$ and a vertex labeling $\varphi_v : V(G) \to \{0, 2, \ldots, 2k_v\}$, in which labeling φ is a total k-labeling of a graph G such that $\varphi(x) = \varphi_v(x)$ if $x \in V(G)$ and $\varphi(x) = \varphi_e(x)$ if $x \in E(G)$, where $k = \max\{k_e, 2k_v\}$. The total k-labeling φ is called an *edge irregular reflexive k-labeling* of the graph G if for every two edges xy, $x'y'$ of the graph G, one has $wt(xy) \neq wt(x'y')$, where $wt(xy) = \varphi_v(x) + \varphi_e(xy) + \varphi_v(y)$. The smallest value of k for which such labeling exists is called a *reflexive edge strength* of the graph G and is denoted by $res(G)$. Bača et al. [1] studied the exact value of the reflexive edge strength for cycles, Cartesian product of two cycles and for join graphs of the path and cycle with $2K_2$. In [5], the authors investigated the exact value of the reflexive edge strength for disjoint union of s isomorphic copies of generalized Peterson graphs. Tanna et al. [8] determined the exact value of the reflexive edge strength for prisms and wheels. Zhang et al. [10] investigated the exact value of the reflexive edge strength for disjoint union of gear graphs and prism graphs. However, a study of the edge irregular reflexive labeling on plane graphs has never been discussed previously.

According to Nishizeki et al. [7], a plane graph is a planar graph which is embedded in the plane, where the planar graph is defined as a graph that is drawn on the plane in such a way that the intersection of any two edges is empty and every edge only incident with its endpoints. In this paper, we extend the study of the papers [6] and [9]. Imran et al. [6] studied a vertex irregular total labeling of four classes of plane graphs and determine its total vertex irregularity strength. While Tarawneh et al. [9] studied an edge irregular labeling of three classes of plane graphs and investigated its edge irregularity strength. Thus, we study the edge irregular reflexive labeling of three classes of plane graphs, namely $\mathcal{C}_n$, $\mathcal{A}_n$ and $\mathcal{B}_n$. Consequently, the exact value of reflexive edge strength of these graphs is obtained.

2 Main Results

The following lemma is required.

Lemma 2.1. [8] *For every graph G,*

$$\operatorname{res}(G) \geq \begin{cases} \left\lceil \frac{|E(G)|}{3} \right\rceil, & \text{if } |E(G)| \not\equiv 2, 3 \pmod 6, \\ \left\lceil \frac{|E(G)|}{3} \right\rceil + 1, & \text{if } |E(G)| \equiv 2, 3 \pmod 6. \end{cases}$$

Let Q_1, Q_2 and Q_3 be three different paths on the vertices $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_{2n}$ and $c_1, c_2, \ldots, c_n$, respectively. Then, $\mathcal{C}_n$ is defined as a graph obtained by the disjoint union $Q_1 \cup Q_2 \cup Q_3$ by adjoining the edges $a_i b_{2i-1}$ and $c_i b_{2i}$, where $i = 1, 2, \ldots, n$. Figure 1 shows a plane graph $\mathcal{C}_n$.

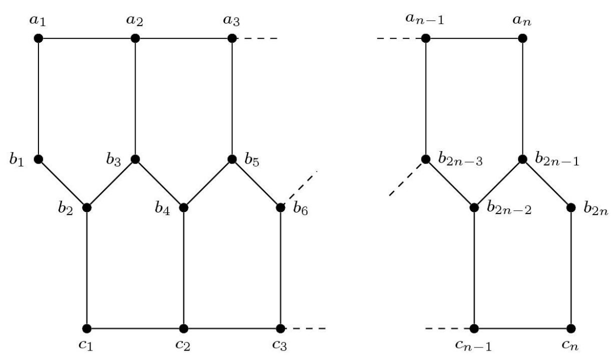

Figure 1: A plane graph $\mathcal{C}_n$, where $n \geq 2$.

The plane graph $\mathcal{C}_n$ consists of a vertex set and an edge set that are defined as $V(\mathcal{C}_n) = \{a_i, c_i : 1 \leq i \leq n\} \cup \{b_i : 1 \leq i \leq 2n\}$ and $E(\mathcal{C}_n) = \{a_i a_{i+1}, c_i c_{i+1} : 1 \leq i \leq n-1\} \cup \{a_i b_{2i-1}, c_i b_{2i} : 1 \leq i \leq n\} \cup \{b_i b_{i+1} : 1 \leq i \leq 2n-1\}$, respectively.

Theorem 2.1. *For $n \geq 2$, $res(\mathcal{C}_n) = 2n$.*

Proof. Since the plane graph $\mathcal{C}_n$ has $6n - 3$ edges, by Lemma 2.1, we have $res(\mathcal{C}_n) \geq k = 2n$. Therefore, $k = 2n$ is the upper bound is proved for the reflexive edge strength of $\mathcal{C}_n$, where $n \geq 2$. Define a total k-labeling φ of $\mathcal{C}_n$ as follows:

$$\varphi(a_i) = 2(n-1), \text{ if } 1 \leq i \leq n.$$

$$\varphi(b_i) = 0, \text{ if } 1 \leq i \leq 2n.$$

$$\varphi(c_i) = 2n, \text{ if } 1 \leq i \leq n.$$

$$\varphi(a_i a_{i+1}) = 3 + i, \text{ if } 1 \leq i \leq n-1.$$

$$\varphi(a_i b_{2i-1}) = 1 + i, \text{ if } 1 \leq i \leq n.$$

$$\varphi(b_i b_{i+1}) = i, \text{ if } 1 \leq i \leq 2n-1.$$

$$\varphi(c_i b_{2i}) = n - 1 + i, \text{ if } 1 \leq i \leq n.$$

$$\varphi(c_i c_{i+1}) = n - 2 + i, \text{ if } 1 \le i \le n-1.$$

Clearly, the maximum vertex label is $k = 2n$. Moreover, the edge labels are at most $2n$ and the maximum is attainable when $n = 2$. Thus, labeling φ is a total k-labeling of $\mathcal{C}_n$. Next, we show the edge weights of $\mathcal{C}_n$ are distinct under the total k-labeling φ.

(i) For $1 \le i \le n-1$,

$$\begin{aligned} wt_\varphi(a_i a_{i+1}) &= \varphi(a_i) + \varphi(a_i a_{i+1}) + \varphi(a_{i+1}) \\ &= 2(n-1) + 3 + i + 2(n-1) \\ &= 4n - 1 + i. \end{aligned}$$

$$\begin{aligned} wt_\varphi(c_i c_{i+1}) &= \varphi(c_i) + \varphi(c_i c_{i+1}) + \varphi(c_{i+1}) \\ &= 2n + n - 2 + i + 2n \\ &= 5n - 2 + i. \end{aligned}$$

(ii) For $1 \le i \le n$,

$$\begin{aligned} wt_\varphi(a_i b_{2i-1}) &= \varphi(a_i) + \varphi(a_i b_{2i-1}) + \varphi(b_{2i-1}) \\ &= 2(n-1) + 1 + i + 0 \\ &= 2n - 1 + i. \end{aligned}$$

$$\begin{aligned} wt_\varphi(c_i b_{2i}) &= \varphi(c_i) + \varphi(c_i b_{2i}) + \varphi(b_{2i}) \\ &= 2n + n - 1 + i + 0 = 3n - 1 + i. \end{aligned}$$

(iii) For $1 \le i \le 2n-1$,

$$wt_\varphi(b_i b_{i+1}) = \varphi(b_i) + \varphi(b_i b_{i+1}) + \varphi(b_{i+1}) = 0 + i + 0 = i.$$

It is easy to verify that the edge weights of $\mathcal{C}_n$ are distinct integers in $\{1, 2, \ldots, 6n-3\}$. The theorem holds. $\square$

The following Figure 2 depicts an example of the corresponding edge irregular reflexive k-labeling of the plane graph $\mathcal{C}_4$.

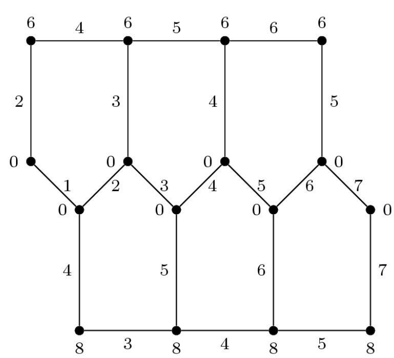

Figure 2: The edge irregular reflexive 8-labeling of the plane graph $\mathcal{C}_4$.

Tarawneh et al. [9] defined $\mathcal{A}_n$ as a plane graph obtained by adding the edges $b_{2i-1}b_{2i+1}$ in the plane graph $\mathcal{C}_n$, where $i = 1, 2, \ldots, n-1$, as shown in Figure 3.

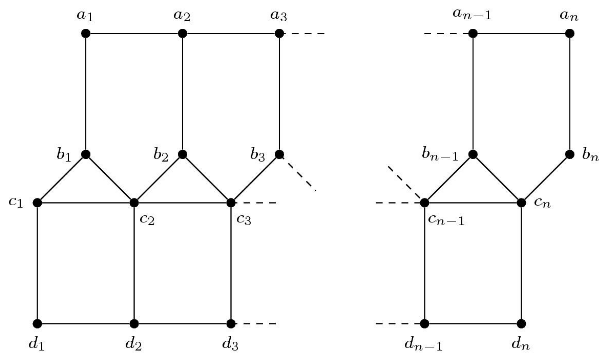

Figure 3: A plane graph $\mathcal{A}_n$, where $n \geq 2$.

The plane graph $\mathcal{A}_n$ consists of a vertex set and an edge set that are defined as $V(\mathcal{A}_n) = \{a_i, b_i, c_i, d_i : 1 \leq i \leq n\}$ and $E(\mathcal{A}_n) = \{a_ia_{i+1}, c_ic_{i+1}, b_ic_{i+1}, d_id_{i+1} : 1 \leq i \leq n-1\} \cup \{a_ib_i, b_ic_i, c_id_i : 1 \leq i \leq n\}$, respectively.

Theorem 2.2. *For $n \geq 2$,*

$$res(\mathcal{A}_n) = \begin{cases} \left\lceil \frac{7n-4}{3} \right\rceil, & \text{if } n \not\equiv 0, 1 \pmod 6, \\ \left\lceil \frac{7n-4}{3} \right\rceil + 1, & \text{if } n \equiv 0, 1 \pmod 6. \end{cases}$$

Proof. The plane graph $\mathcal{A}_n$ has $7n-4$ edges. According to Lemma 2.1, we have

$$res(\mathcal{A}_n) \geq k = \begin{cases} \left\lceil \frac{7n-4}{3} \right\rceil, & \text{if } n \not\equiv 0, 1 \pmod 6, \\ \left\lceil \frac{7n-4}{3} \right\rceil + 1, & \text{if } n \equiv 0, 1 \pmod 6. \end{cases}$$

We now prove that k is the upper bound for the reflexive edge strength of $\mathcal{A}_n$, where $n \geq 2$. Define a total k-labeling φ of $\mathcal{A}_n$ as follows:

For $1 \leq i \leq n$,

$$\varphi(a_i) = \varphi(b_i) = 0,$$

$$\varphi(c_i) = 2(n-1),$$

$$\varphi(d_i) = \begin{cases} k-1, & \text{if } n \equiv 5 \pmod 6, \\ k, & \text{if } n \not\equiv 5 \pmod 6. \end{cases}$$

The edges are labeled in the following ways.

(i) For $1 \leq i \leq n-1$,

$$\varphi(a_ia_{i+1}) = i,$$

$$\varphi(b_ic_{i+1}) = 1 + 2i,$$

$$\varphi(c_ic_{i+1}) = 2+i,$$

$$\varphi(d_id_{i+1}) = \begin{cases} 2(3n-k)-1+i, & \text{if } n \equiv 5 \pmod 6, \\ 2(3n-k)-3+i, & \text{if } n \not\equiv 5 \pmod 6. \end{cases}$$

(ii) For $1 \le i \le n$,

$$\varphi(a_ib_i) = n-1+i,$$

$$\varphi(b_ic_i) = 2i,$$

$$\varphi(c_id_i) = \begin{cases} 3n-k+i, & \text{if } n \equiv 5 \pmod 6, \\ 3n-k-1+i, & \text{if } n \not\equiv 5 \pmod 6. \end{cases}$$

It can be seen that, (a) for $n = 2, 3, 4, 10$, the maximum vertex label is $k = \lceil \frac{7n-4}{3} \rceil$ which is greater than or equal to the edge labels; (b) for $n \equiv 5 \pmod 6$, the maximum edge label is $k = \lceil \frac{7n-4}{3} \rceil$ which is greater than all vertex labels; and (c) otherwise, the maximum vertex label is k which is greater than all edge labels, such that $k = \lceil \frac{7n-4}{3} \rceil + 1$ for $n \equiv 0, 1 \pmod 6$ or $k = \lceil \frac{7n-4}{3} \rceil$ for $n \neq 2, 3, 4, 10$ and $n \not\equiv 0, 1, 5 \pmod 6$. Thus, labeling φ is a total k-labeling. Next, we show the edge weights of $\mathcal{A}_n$ are distinct under the total k-labeling φ.

(i) For $1 \le i \le n-1$,

$$wt_\varphi(a_ia_{i+1}) = \varphi(a_i) + \varphi(a_ia_{i+1}) + \varphi(a_{i+1}) = 0+i+0 = i.$$

$$\begin{aligned} wt_\varphi(b_ic_{i+1}) &= \varphi(b_i) + \varphi(b_ic_{i+1}) + \varphi(c_{i+1}) \\ &= 0+1+2i+2(n-1) \\ &= 2(n-1+i)+1. \end{aligned}$$

$$\begin{aligned} wt_\varphi(c_ic_{i+1}) &= \varphi(c_i) + \varphi(c_ic_{i+1}) + \varphi(c_{i+1}) \\ &= 2(n-1)+2+i+2(n-1) \\ &= 2(2n-1)+i. \end{aligned}$$

When $n \equiv 5 \pmod 6$,

$$\begin{aligned} wt_\varphi(d_id_{i+1}) &= \varphi(d_i) + \varphi(d_id_{i+1}) + \varphi(d_{i+1}) \\ &= k-1+2(3n-k)-1+i+k-1 \\ &= 3(2n-1)+i. \end{aligned}$$

When $n \not\equiv 5 \pmod 6$,

$$\begin{aligned} wt_\varphi(d_id_{i+1}) &= \varphi(d_i) + \varphi(d_id_{i+1}) + \varphi(d_{i+1}) \\ &= k+2(3n-k)-3+i+k \\ &= 3(2n-1)+i. \end{aligned}$$

(ii) For $1 \le i \le n$,

$$wt_\varphi(a_ib_i) = \varphi(a_i) + \varphi(a_ib_i) + \varphi(b_i) = 0+n-1+1+0 = n-1+i.$$

$$wt_\varphi(b_ic_i) = \varphi(b_i) + \varphi(b_ic_i) + \varphi(c_i) = 0+2i+2(n-1) = 2(n-1+i).$$

When $n \equiv 5 \pmod 6$,

$$\begin{aligned} wt_\varphi(c_i d_i) &= \varphi(c_i) + \varphi(c_i d_i) + \varphi(d_i) \\ &= 2(n-1) + 3n - k + i + k - 1 \\ &= 5n - 3 + i. \end{aligned}$$

When $n \not\equiv 5 \pmod 6$,

$$\begin{aligned} wt_\varphi(c_i d_i) &= \varphi(c_i) + \varphi(c_i d_i) + \varphi(d_i) \\ &= 2(n-1) + 3n - k - 1 + i + k \\ &= 5n - 3 + i. \end{aligned}$$

It is easy to verify that the edge weights of $\mathcal{A}_n$ are distinct integers in $\{1, 2, \ldots, 7n-4\}$. The theorem holds. □

The following Figure 4 illustrates the corresponding edge irregular reflexive k-labeling of (a) the plane graph $\mathcal{A}_5$ and (b) the plane graph $\mathcal{A}_6$.

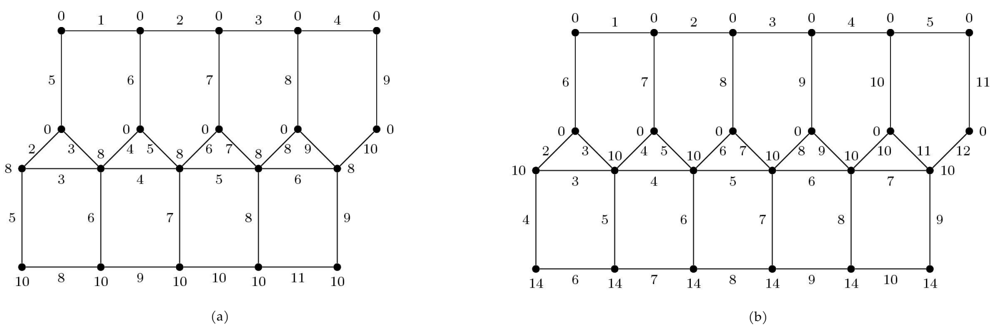

Figure 4: (**a**) The edge irregular reflexive 11-labeling of plane graph $\mathcal{A}_5$. (**b**) The edge irregular reflexive 14-labeling of plane graph $\mathcal{A}_6$.

Let P_1, P_2, P_3 and P_4 be four distinct paths on the vertices $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$, $c_1, c_2, \ldots, c_n$, and $d_1, d_2, \ldots, d_n$, respectively. Then, the plane graph $\mathcal{B}_n$ is defined as the disjoint union $P_1 \cup P_2 \cup P_3 \cup P_4$ by adjoining the edges a_ib_i, b_ic_i, c_id_i for $1 \le i \le n$ and the edges a_ib_{i+1}, d_ic_{i+1} for $1 \le i \le n-1$, as shown in Figure 5.

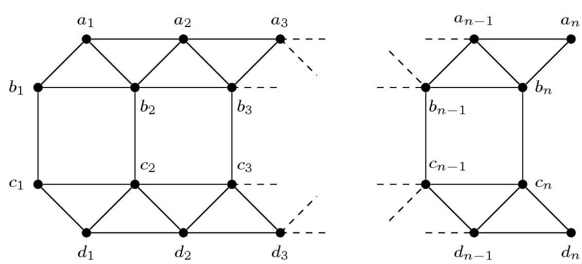

Figure 5: A plane graph $\mathcal{B}_n$, where $n \ge 2$.

The vertex set and edge set of the plane graph $\mathcal{B}_n$ are defined as $V(\mathcal{B}_n) = \{a_i, b_i, c_i, d_i : 1 \le i \le n\}$ and $E(\mathcal{B}_n) = \{a_ia_{i+1}, b_ib_{i+1}, c_ic_{i+1}, d_id_{i+1}, a_ib_{i+1}, d_ic_{i+1} : 1 \le i \le n-1\} \cup \{a_ib_i, b_ic_i, c_id_i : 1 \le i \le n\}$, respectively.

Theorem 2.3. *For $n \geq 2$,*

$$res(\mathcal{B}_n) = \begin{cases} 3n-2, & \textit{if } n \equiv 0\,(\textit{mod}\,2), \\ 3n-1, & \textit{if } n \equiv 1\,(\textit{mod}\,2). \end{cases}$$

Proof. Since the plane graph $\mathcal{B}_n$ has $9n-6$ edges, by Lemma 2.1, we have

$$res(\mathcal{B}_n) \geq k = \begin{cases} 3n-2, & \text{if } n \equiv 0\,(\text{mod}\,2), \\ 3n-1, & \text{if } n \equiv 1\,(\text{mod}\,2). \end{cases}$$

We show that k is the upper bound for the reflexive edge strength of $\mathcal{B}_n$, where $n \geq 2$. First, two cases are distinguished according to the parity of n.

Case 1. n is even. Suppose $n = 2$, $res(\mathcal{B}_2) \geq 4$ is obtained where the vertices can only be labeled with $0's$, $2's$ and $4's$. The corresponding labelings are (a) for $i = 1, 2$, the vertices are labeled as $\varphi(a_i) = 0$, $\varphi(b_i) = 2(i-1)$, $\varphi(c_i) = 2i$ and $\varphi(d_i) = 4$; and (b) the edges are labeled as $\varphi(a_1a_2) = 1$, $\varphi(a_1b_2) = 3$, $\varphi(b_1b_2) = 1$, $\varphi(c_1c_2) = 2$, $\varphi(d_1c_2) = 2$ and $\varphi(d_1d_2) = 4$, whereas $\varphi(a_ib_i) = 2i$, $\varphi(b_ic_i) = 1+i$ and $\varphi(c_id_i) = 2i-1$ for $i = 1, 2$. It is easy to check that $\mathcal{B}_2$ admits an edge irregular reflexive 4-labeling as required. Next, a total k-labeling φ of $\mathcal{B}_n$ for $n \geq 4$ is defined as follows:

(i) For $1 \leq i \leq n$,

$$\varphi(a_i) = 0,$$
$$\varphi(b_i) = n-2,$$
$$\varphi(c_i) = 2n,$$
$$\varphi(d_i) = k,$$
$$\varphi(a_ib_i) = 2i,$$
$$\varphi(b_ic_i) = n-1+i,$$
$$\varphi(c_id_i) = n-3+2i.$$

(ii) For $1 \leq i \leq n-1$,

$$\varphi(a_ia_{i+1}) = i,$$
$$\varphi(a_ib_{i+1}) = 1+2i,$$
$$\varphi(b_ib_{i+1}) = n+2+i,$$
$$\varphi(c_ic_{i+1}) = n-3+i,$$
$$\varphi(d_ic_{i+1}) = n-2+2i,$$
$$\varphi(d_id_{i+1}) = 2n-1+i.$$

Under the labeling φ, maximum vertex label is $k = 3n-2$ which is greater than or equal to the edge labels. Thus, labeling φ is a total k-labeling of $\mathcal{B}_n$. Next, the edge weights of $\mathcal{B}_n$ is computed.

(i) For $1 \leq i \leq n-1$,

$$wt_{\varphi}(a_i a_{i+1}) = \varphi(a_i) + \varphi(a_i a_{i+1}) + \varphi(a_{i+1}) = 0 + i + 0 = i.$$

$$\begin{aligned} wt_{\varphi}(a_i b_{i+1}) &= \varphi(a_i) + \varphi(a_i b_{i+1}) + \varphi(b_{i+1}) \\ &= 0 + 2i + 1 + n - 2 \\ &= n - 1 + 2i. \end{aligned}$$

$$\begin{aligned} wt_{\varphi}(b_i b_{i+1}) &= \varphi(b_i) + \varphi(b_i b_{i+1}) + \varphi(b_{i+1}) \\ &= n - 2 + n + 2 + i + n - 2 \\ &= 3n - 2 + i. \end{aligned}$$

$$\begin{aligned} wt_{\varphi}(c_i c_{i+1}) &= \varphi(c_i) + \varphi(c_i c_{i+1}) + \varphi(c_{i+1}) \\ &= 2n + n - 3 + i + 2n \\ &= 5n - 3 + i. \end{aligned}$$

$$\begin{aligned} wt_{\varphi}(d_i c_{i+1}) &= \varphi(d_i) + \varphi(d_i c_{i+1}) + \varphi(c_{i+1}) \\ &= k + n - 2 + 2i + 2n \\ &= 2(3n - 2 + i). \end{aligned}$$

$$\begin{aligned} wt_{\varphi}(d_i d_{i+1}) &= \varphi(d_i) + \varphi(d_i d_{i+1}) + \varphi(d_{i+1}) \\ &= k + 2n - 1 + i + k \\ &= 8n - 5 + i. \end{aligned}$$

(ii) For $1 \leq i \leq n$,

$$wt_{\varphi}(a_i b_i) = \varphi(a_i) + \varphi(a_i b_i) + \varphi(b_i) = 0 + 2i + n - 2 = n - 2 + 2i.$$

$$\begin{aligned} wt_{\varphi}(b_i c_i) &= \varphi(b_i) + \varphi(b_i c_i) + \varphi(c_i) \\ &= n - 2 + n - 1 + i + 2n \\ &= 4n - 3 + i. \end{aligned}$$

$$\begin{aligned} wt_{\varphi}(c_i d_i) &= \varphi(c_i) + \varphi(c_i d_i) + \varphi(d_i) \\ &= 2n + n - 3 + 2i + k \\ &= 6n - 5 + 2i. \end{aligned}$$

It clearly shows that all the edge weights of $\mathcal{B}_n$ are distinct integers in $\{1, 2, \ldots, 9n - 6\}$.
Case 2. n is odd. Define a total k-labeling φ as follows:

(i) For $1 \leq i \leq n$,

$$\varphi(a_i) = 0,$$
$$\varphi(b_i) = n - 1,$$
$$\varphi(c_i) = 2n,$$
$$\varphi(d_i) = k,$$
$$\varphi(a_i b_i) = 2i - 1,$$
$$\varphi(b_i c_i) = n - 2 + i,$$
$$\varphi(c_i d_i) = n - 4 + 2i.$$

(ii) For $1 \leq i \leq n-1$,

$$\begin{aligned}
\varphi(a_i a_{i+1}) &= i, \\
\varphi(a_i b_{i+1}) &= 2i, \\
\varphi(b_i b_{i+1}) &= n+i, \\
\varphi(c_i c_{i+1}) &= n-3+i, \\
\varphi(d_i c_{i+1}) &= n-3+2i, \\
\varphi(d_i d_{i+1}) &= 2n-3+i.
\end{aligned}$$

It definitely shows that the maximum vertex label is greater than all edge labels, where the maximum vertex label is $k = 3n-1$. Thus, labeling φ is a total k-labeling of $\mathcal{B}_n$. Next, we compute the edge weights of $\mathcal{B}_n$ as follows:

(i) For $1 \leq i \leq n-1$,

$$wt_\varphi(a_i a_{i+1}) = \varphi(a_i) + \varphi(a_i a_{i+1}) + \varphi(a_{i+1}) = 0 + i + 0 = i.$$

$$wt_\varphi(a_i b_{i+1}) = \varphi(a_i) + \varphi(a_i b_{i+1}) + \varphi(b_{i+1}) = 0 + 2i + n - 1 = n - 1 + 2i.$$

$$\begin{aligned}
wt_\varphi(b_i b_{i+1}) &= \varphi(b_i) + \varphi(b_i b_{i+1}) + \varphi(b_{i+1}) \\
&= n-1+n+i+n-1 \\
&= 3n-2+i. \\
wt_\varphi(c_i c_{i+1}) &= \varphi(c_i) + \varphi(c_i c_{i+1}) + \varphi(c_{i+1}) \\
&= 2n+n-3+i+2n \\
&= 5n-3+i. \\
wt_\varphi(d_i c_{i+1}) &= \varphi(d_i) + \varphi(d_i c_{i+1}) + \varphi(c_{i+1}) \\
&= k+n-3+2i+2n \\
&= 2(3n-2+i). \\
wt_\varphi(d_i d_{i+1}) &= \varphi(d_i) + \varphi(d_i d_{i+1}) + \varphi(d_{i+1}) \\
&= k+2n-3+i+k \\
&= 8n-5+i.
\end{aligned}$$

(ii) For $1 \leq i \leq n$,

$$wt_\varphi(a_i b_i) = \varphi(a_i) + \varphi(a_i b_i) + \varphi(b_i) = 0 + 2i - 1 + n - 1 = n - 2 + 2i.$$

$$\begin{aligned}
wt_\varphi(b_i c_i) &= \varphi(b_i) + \varphi(b_i c_i) + \varphi(c_i) \\
&= n-1+n-2+i+2n \\
&= 4n-3+i. \\
wt_\varphi(c_i d_i) &= \varphi(c_i) + \varphi(c_i d_i) + \varphi(d_i) \\
&= 2n+n-4+2i+k \\
&= 6n-5+2i.
\end{aligned}$$

It is easy to verify that the edge weights of $\mathcal{B}_n$ are distinct integers in $\{1, 2, \ldots, 9n-6\}$. The theorem holds.

□

The following Figure 6 shows the corresponding edge irregular reflexive k-labeling of (a) the plane graph $\mathcal{B}_4$ and (b) the plane graph $\mathcal{B}_5$.

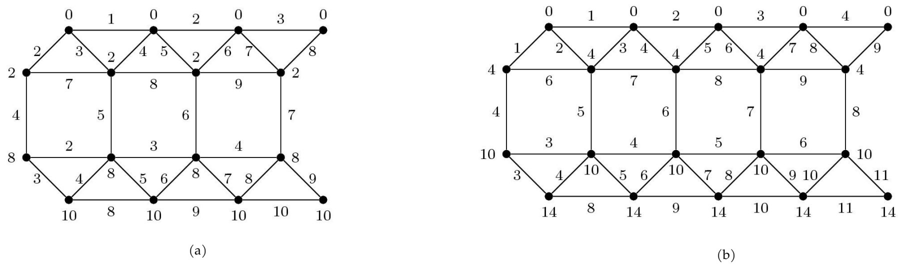

Figure 6: (**a**) The edge irregular reflexive 10-labeling of plane graph $\mathcal{B}_4$. (**b**) The edge irregular reflexive 14-labeling of plane graph $\mathcal{B}_5$.

3 Conclusions

In this paper, the reflexive edge strength of the plane graph $\mathcal{C}_n$, the plane graph $\mathcal{A}_n$, and the plane graph $\mathcal{B}_n$ are successfully determined via Theorems 2.1, 2.2, and 2.3, respectively. Moreover, these results provided further support to the following conjecture.

Conjecture 3.1. [5, 10] *Any graph G with maximum degree $\Delta(G)$ satisfies:*

$$res(G) = \max\left\{\left\lfloor \frac{\Delta+2}{2}\right\rfloor, \left\lceil \frac{|E(G)|}{3}\right\rceil + r\right\},$$

where $r = 1$ for $|E(G)| \equiv 2, 3 (\text{mod } 6)$, and zero otherwise.

Note that the graphs presented in this extensive study are restricted, only three classes of plane graphs. Nonetheless, this extensive study clearly showed the reflexive edge strength of these classes of plane graphs, which is different from the studies of total vertex irregularity strength in [6] and the edge irregularity strength in [9]. As to our understanding, to study the edge irregular reflexive labeling on general graphs, or even on planar graphs is an enormous challenge. However, this present work is the first study of the edge irregular reflexive labeling on plane graphs, which provides a great potential of extensive study in this area. We suggest to study the following problems: the edge irregular reflexive labeling on the families of plane graphs, such as outerplanar graphs, maximal planar graphs, Halin graphs, planar n-trees and families of convex polytopes.

Acknowledgement The authors wish to thank the referees for their valuable comments. This research is supported by the Fundamental Research Grant Scheme (FRGS), Ministry of Higher Education, Malaysia with Reference Number FRGS/1/2020/STG06/UMT/02/1 (Grant Vot. 59609).

References

[1] M. Bača, M. Irfan, J. Ryan, A. Semaničová-Feňovčíková & D. Tanna (2019). Note on edge irregular reflexive labelings of graphs. *AKCE International Journal of Graphs and Combinatorics*, *16*(2), 145–157.

[2] M. Bača, S. Jendrol', M. Miller & J. Ryan (2007). On irregular total labelings. *Discrete Mathematics*, *307*(11-12), 1378–1388.

[3] G. Chartrand, M. S. Jacobson, J. Lehel, O. R. Oellermann, S. Ruiz & F. Saba (1988). Irregular networks. *Congressus Numerantium*, *64*, 197–210.

[4] J. A. Gallian (2021). A dynamic survey of graph labeling. *The Electronic Journal of Combinatorics*, pp. 1–576. https://doi.org/10.37236/27.

[5] J. L. G. Guirao, S. Ahmad, M. K. Siddiqui & M. Ibrahim (2018). Edge irregular reflexive labeling for the disjoint union of generalized Petersen graph. *Mathematics*, *6*, 304. https://doi:10.3390/math6120304.

[6] M. Imran, S. A. Bokhary & A. Ahmad (2015). Total vertex irregularity strength of grid-like-plane graphs. *Science International-Lahore*, *27*(2), 821–828.

[7] T. Nishizeki & N. Chiba (1988). *Planar graphs: Theory and algorithms*. North-Holland Mathematics Studies, Netherlands.

[8] D. Tanna, J. Ryan & A. Semaničová-Feňovčíková (2017). Edge irregular reflexive labeling of prisms and wheels. *Australasian Journal of Combinatorics*, *69*(3), 394–401.

[9] I. Tarawneh, R. Hasni, A. Ahmad & M. A. Asim (2021). On the edge irregularity strength for some classes of plane graphs. *AIMS Mathematics*, *6*(3), 2724–2731.

[10] X. Zhang, M. Ibrahim, S. A. H. Bokhary & M. K. Siddiqui (2018). Edge irregular reflexive labeling for the disjoint union of gear graphs and prism graphs. *Mathematics*, *6*, 142. https://doi.org/10.3390/math6090142.