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Trigonometrically-Fitted Fifth Order Four-Step Predictor-Corrector Method for Solving Linear Ordinary Differential Equations with Oscillatory Solutions

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Abstract

In this paper, we proposed a trigonometrically-fitted fifth order four-step predictor-corrector method based on the four-step Adams-Bashforth method as predictor and five-step Adams-Moulton method as corrector to solve linear ordinary differential equations with oscillatory solutions. This method is constructed which exactly integrate initial value problems whose solutions can be expressed as linear combinations of the set functions $\{\sin(vx), \cos(vx)\}$ with $v \in$ R, where v represents an approximation of the frequency of the problem. The frequency will be used in the method to raise the accuracy of the solution. Stability of the proposed method is examined and the corresponding region of stability is depicted. The new fifth algebraic order trigonometrically-fitted predictor-corrector method is applied to solve the initial value problems whose solutions involved trigonometric functions. Numerical results presented proved that the prospective method is more efficient than the widely used methods for the numerical solution of linear ordinary differential equations with oscillating solutions.

Keywords: trigonometrically-fitted; fifth order four-step Predictor-Corrector method; Adams-Bashforth-Moulton method; LODEs; oscillatory solutions.

1 Introduction

Equation in the form:

$$y''(x) = f(x, y), \tag{1}$$

is used as mathematical models for problems in celestial mechanics, physical chemistry, chemical physics, quantum mechanics, material sciences and many more fields. The above equation which is usually linear, have oscillatory or periodic solutions and deserves special attention (see [6, 8]). The focus of the numerical solution of the above equation is the subject of extensive research activity over the last two decades (see [14, 7, 22, 20, 19, 21, 3, 13, 17]). Extensive reviews of the methods developed for the solution of (1) with oscillating behavior can be found in [19, 21] and the references therein, Ibrahim and Ikhile [7], as well as [1, 13, 17].

To obtain a more accurate numerical results for problem (1), researchers have derived methods which take into account the nature of the problems to be solved. To do so they used fitting techniques such as phase-fitting, exponential and trigonometric fitting to enhance the efficiency of the original method. By doing so more accurate numerical results for highly oscillatory problems can be obtained. The exponential and trigonometric fitting methodology first proposed by Lyche [12] is one of the best techniques for designing efficient methods for solving linear first order initial value problems (IVPs) with periodic solutions. Shokri and et al. [20] developed a new family of multiderivative methods with vanishing phase-lag for the numerical integration of IVPs with oscillating solutions. More recently, [3] introduced phase-fitting for finite difference process for solving IVPs arising in chemistry. Trigonometric fitting technique have also been applied to other types of predictor-corrector (P-C) methods, such as the work in (see [11]), where they developed a general class of trigonometrically-fitted two-step hybrid (TFTSH) method for solving second order IVPs. The original non-trigonometric form of the method was introduced in [2].

In this work, the fifth order Adams P-C method is trigonometrically-fitted and used to solve second order linear ordinary differential equations (LODEs) with oscillatory solutions, by reducing the problems to a system of first order IVPs. This paper has been structured as follows: the derivation of the trigonometric fitting method is given in Section 2. Section 3 is focused on the stability of the new scheme. The numerical illustrations are given in Section 4, followed by concluding remarks in Section 5.

2 Trigonometrically-fitted fifth algebraic order (P-C) method

The P-C family of methods below has been widely used in obtaining numerical solutions of first order ODEs see [18]. The methods can be written as,

$$\begin{aligned} \bar{y}_{n+1} &= y_n + h \sum_{i=0}^{k-1} b_i \nabla^i f_n \ , \\ y_{n+1} &= y_n + h \sum_{i=0}^{k} \beta_i \nabla^i \bar{f}_{n+1}. \end{aligned} \tag{2}$$

In (2) the corrector is always one order higher than the predictor and the total algebraic order of the system is determined by the corrector's order. In the general case (2), after expressing the backward differences in terms of f_{n-i}. The following fifth algebraic order four-step scheme can be obtained,

$$\begin{aligned} \bar{y}_{n+1} &= y_n + h\left(a_0 f_n + a_1 f_{n-1} + a_2 f_{n-2} + a_3 f_{n-3}\right), \\ y_{n+1} &= y_n + h(c_0 \bar{f}_{n+1} + c_1 f_n + c_2 f_{n-1} + c_3 f_{n-2} + c_4 f_{n-3}), \end{aligned} \tag{3}$$

where, in terms of $f_{n-i}, a_i,\ i = 0, 1, 2, 3$, are known coefficients of Adams-Bashforth and the coefficients $c_i,\ i = 0, 1, 2, 3, 4$, correspond to the Adams-Moulton coefficients for (2) above.

In order to ensure the accuracy of the method (3) for any linear combination of functions:

$$\left\{1, x, x^2, \cos(\pm vx), \sin(\pm vx)\right\}, \tag{4}$$

the following system of equations must remain in place:

$$\begin{aligned} \cos(\omega) = 1 - c_0 a_0 \omega^2 - c_0 a_1 \omega^2 \cos(\omega) \\ -2c_0 a_2 \omega^2 \cos(\omega)^2 \\ +c_0 a_2 \omega^2 - 4c_0 a_3 \omega^2 \cos(\omega)^3 \\ +3c_0 a_3 \omega^2 \cos(\omega) \\ +c_2 \omega \sin(\omega) \\ -c_4 \omega \sin(\omega) \\ +2c_3 \omega \sin(\omega) \cos(\omega) \\ +4c_4 \omega \sin(\omega) \cos(\omega)^2, \end{aligned} \tag{5}$$

$$\begin{aligned} \sin(\omega) = \omega(4\cos(\omega)^2 \sin(\omega) \omega a_3 c_0 \\ +2\cos(\omega) \sin(\omega) \omega a_2 c_0 + 4\cos(\omega)^3 c_4 \\ + \sin(\omega) \omega a_1 c_0 + \sin(\omega) \omega a_3 c_0 \\ +2\cos(\omega)^2 c_3 + \cos(\omega) c_2 - 3\cos(\omega) c_4 + c_0 + c_1 - c_3), \end{aligned} \tag{6}$$

where $\omega = vh$, it is noted here that in the above system of equations (5) and (6) are derived from the requirement that method (3) be accurate for any linear combination of functions, $\{\cos(\pm vx), \sin(\pm vx)\}$.

The known coefficients of Adams-Bashforth in terms of f_{n-i}.

$$a_0 = \frac{55}{24},\ a_1 = -\frac{59}{24},\ a_2 = \frac{37}{24},\ a_3 = -\frac{9}{24},\ c_0 = \frac{251}{720},\ c_1 = \frac{646}{720},\ c_2 = -\frac{264}{720}\ . \tag{7}$$

Substituting the coefficients into (5) and (6) and solving for c_3 and c_4 we have,

$$
\begin{aligned}
c_3 = -\frac{1}{\sin(\omega)\omega}\Big(&\frac{9287}{2160}\sin(\omega)^2\cos(\omega)^3\omega^2 - \\
&\frac{14809}{4320}\sin(\omega)^2\cos(\omega)^2\omega^2 - \frac{9287}{8640}\sin(\omega)^2\cos(\omega)\omega^2 - \\
&\frac{251}{120}\sin(\omega)^2\cos(\omega)^4\omega^2 - \frac{11}{15}\sin(\omega)\cos(\omega)\omega + \\
&\frac{299}{60}\sin(\omega)\cos(\omega)^2\omega + \frac{4267}{4320}\sin(\omega)^2\omega^2 + \frac{251}{180}\cos(\omega)^2\omega^2 - \\
&\frac{251}{120}\cos(\omega)^6\omega^2 + \frac{9287}{2160}\cos(\omega)^5\omega^2 - \frac{251}{864}\cos(\omega)^4\omega^2 - \\
&\frac{1255}{576}\cos(\omega)^3\omega^2 - \frac{299}{240}\sin(\omega)\omega - \frac{251}{320}\cos(\omega)\omega^2 + \\
&4\cos(\omega)^4 - 4\sin(\omega)^2\cos(\omega)^2 + \sin(\omega)^2 - 4\cos(\omega)^3 + \\
&3\cos(\omega) - 3\cos(\omega)^2\Big),
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
c_4 = \frac{1}{\sin(\omega)\omega}\Big(&-\frac{251}{240}\sin(\omega)^2\cos(\omega)^3\omega^2 - \\
&\frac{251}{240}\cos(\omega)^5\omega^2 + \frac{9287}{4320}\sin(\omega)^2\cos(\omega)^2\omega^2 + \frac{9287}{4320}\cos(\omega)^4\omega^2 - \\
&\frac{4267}{2160}\sin(\omega)^2\cos(\omega)\omega^2 - \frac{1757}{4320}\cos(\omega)^3\omega^2 - \\
&\frac{4769}{8640}\cos(\omega)^2\omega^2 + \frac{251}{540}\cos(\omega)\omega^2 + \frac{299}{120}\sin(\omega)\cos(\omega)\omega - \\
&\frac{251}{960}\omega^2 - 2\sin(\omega)^2\cos(\omega) - \frac{11}{30}\sin(\omega)\omega + 2\cos(\omega)^3 - \\
&2\cos(\omega)^2 - \cos(\omega) + 1\Big).
\end{aligned} \tag{9}
$$

To avoid heavy cancellation in the implementation, the following expansions of the Taylor series should be used,

$$
\begin{aligned}
c_3 = \frac{53}{360} + \frac{251}{320}\omega^2 - \frac{556649}{518400}\omega^4 + \frac{1518367}{4354560}\omega^6 - \frac{18400003}{290304000}\omega^8 + \\
\frac{1449609563}{172440576000}\omega^{10} - \frac{705712460059}{941525544960000}\omega^{12} + \dots
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
c_4 = -\frac{19}{720} - \frac{251}{480}\omega^2 + \frac{127439}{518400}\omega^4 - \frac{6161}{194400}\omega^6 + \frac{192829}{41472000}\omega^8 - \\
\frac{19496737}{86220288000}\omega^{10} + \frac{14517700489}{941525544960000}\omega^{12} + \dots
\end{aligned} \tag{11}
$$

The local truncation error of the above method is given by,

$$
L.T.E = h^6\left(0.34868y_n^{(6)} - 0.031692y_n^{(5)} + 0.36667\omega^2 y_n^{(4)}\right) + O(h^7), \tag{12}
$$

where $y_n^{(4)}$ is the fourth derivative of y at x_n, $y_n^{(5)}$ is the fifth derivative of y at x_n, and $y_n^{(6)}$ is the sixth derivative of y at x_n. We note here that in order to produce Eq. (12) we express the quantities $y_{n+1}, y_{n-1}, y_{n-2}, y_{n-3}$ and $f_{n+1}, f_{n-1}, f_{n-2}, f_{n-3}$ around the point x_n and then we substitute the expressions into Eq. (3).

Since $\omega = vh$, it can be seen that the trigonometric fitted method becomes the original P$-$C method for the corresponding algebraic order and step number when $v = 0$.

3 Stability analysis

Applying scheme (3) with the coefficients $a_0 = \frac{55}{24}$, $a_1 = -\frac{59}{24}$, $a_2 = \frac{37}{24}$, $a_3 = -\frac{9}{24}$, $c_0 = \frac{251}{720}$, $c_1 = \frac{646}{720}$, $c_2 = -\frac{264}{720}$, to the scale test equation,

$$y' = \lambda y, \tag{13}$$

and taking $H = \lambda h$, the following difference equation is obtained,

$$y_{n+1} - A(H)\, y_n + B(H)\, y_{n-1} + C(H)\, y_{n-2} + D(H)\, y_{n-3} = 0, \tag{14}$$

where,

$$A(H) = 1 + \frac{299}{240}H + \frac{2761}{3456}H^2, \tag{15}$$

$$B(H) = -\frac{14809}{17280}H^2 + \frac{11}{30}H, \tag{16}$$

$$C(H) = \frac{9287}{17280}H^2 - Hc_3, \tag{17}$$

$$D(H) = -\frac{251}{1920}H^2 - Hc_4. \tag{18}$$

The characteristic equation of (14) shall be given by,

$$r^4 - A(H)\, r^3 + B(H)\, r^2 + C(H)\, r + D(H) = 0. \tag{19}$$

Solving the above equation in H using the boundary locus technique [9] and substituting $r = \exp(i\theta)$, where $i = \sqrt{-1}$, we can plot the absolute stability regions for $0 \leq \theta \leq 2\pi$. Fig. 1 shows the region of absolute stability for the original case and Fig. 2 shows the regions of absolute stability for the trigonometric fitted case.

As we can see in Fig. 2 the larger the frequency υ, the larger is the absolute stability region. As a matter of fact, it appears that our trigonometric fitted scheme has enormous gains in absolute stability. Such very large regions of absolute stability place our scheme in a highly advantageous position and could therefore be used to solve a much wider range of problems effectively. In contrast to other comparable methods with much smaller stability regions. Among other things, it remains to be investigated, for example, until how large υ, the region of absolute stability continues to grow.

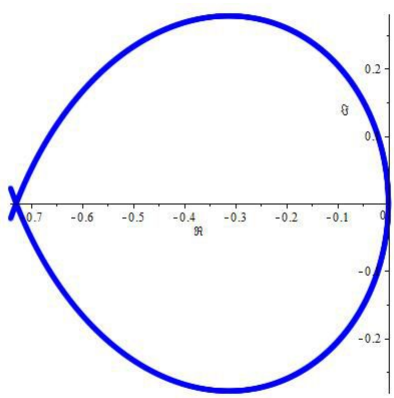

Figure 1: The Absolute Stability Region.

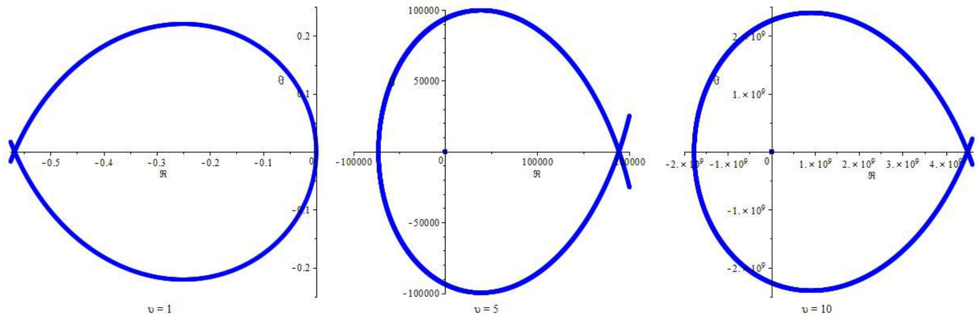

Figure 2: The Stability Region with Trigonometric-fitted.

4 Numerical results

In this section, the new method is applied to the numerical solution of three problems. The first is the inhomogeneous equation studied by [15]. The second is a high-frequency Inhomogeneous system studied by [10] and the third is a high-frequency Inhomogeneous problem studied by [5]. The problems are first reduced to a system of first order ODEs. The test problems are listed below and the integration interval is [0, 1000].

Problem 1: [15]

$$y'' = -100\,y + 99\ \sin(t),\ y\,(0) = 1,\ y'\,(0) = 11,$$

the exact solution is

$$y = \cos\left(10\,t\right) + \sin\left(10\,t\right) + \sin\left(t\right), \upsilon = 10.$$

Problem 2: [10]

$$y_1{''} = -13\,y_1 + 12\,y_2 + 9\ \cos\left(2\,t\right) - 12\ \sin\left(2\,t\right), y_1\ (0) = 1,\ y_1{'}\,(0) = -4,$$
$$y_2{''} = 12\,y_1 - 13\,y_2 - 12\ \cos\left(2\,t\right) + 9\ \sin(2\,t),\ y_2\,(0) = 0,\ y_2{'}\,(0) = 8,$$

the exact solution is

$$y_1 = \sin\left(t\right) - \sin(5\,t) +\ \cos\left(2\,t\right)$$
$$y_2 = \sin\left(t\right) + \sin\left(5\,t\right) + \sin\left(2\,t\right), \upsilon = 5.$$

Problem 3: [5]

$$y_1{''} = -400\,y_1 + 400\,e^{-0.05\ t} + 0.0025\,e^{-0.05\ t},\ y_1\,(0) = 1.1,\ y_1'\,(0) = -0.05,$$
$$y_2{''} = -400\,y_2 + 400\,e^{-0.05\ t} + 0.0025\,e^{-0.05\ t},\ y_2\,(0) = 1.0,\ y_2'\,(0) = 1.95,$$

the exact solution is

$$y_1 = 0.1\ \cos\left(20\,t\right) + e^{-0.05\ t}$$
$$y_2 = 0.1\ \sin\left(20\,t\right) + e^{-0.05\ t}, \upsilon = 20.$$

In all our numerical illustrations, the following methods are compared and the following notations are used,

- TF4SPC - The new trigonometrically fitted fifth order four-step predictor-corrector method.
- 4SPC - The original fifth order P-C method specified by (3) that is the method without trigonometric-fitting.
- TF3SPC - Fourth order trigonometrically fitted three-step predictor-corrector method in [16].
- EmbDP4(5) - Fifth order embedded Runge-Kutta Dormand-Prince 5(4) method in [4].

Table 1: Comparison between the proposed method and the other current methods for Problem 1.

STEP SIZE(H)	**METHOD**	**MAX ERRORS**
0.0025	TF4SPC	2.3073e-04
	4SPC	3.7807e-03
	TF3SPC	3.0605e-03
	EmbDP4(5)	3.3496e-02
0.00125	TF4SPC	5.7653e-05
	4SPC	3.8160e-04
	TF3SPC	8.9456e-03
	EmbDP4(5)	9.6163e-02
0.001	TF4SPC	3.6917e-06
	4SPC	4.9834e-05
	TF3SPC	5.2224e-04
	EmbDP4(5)	4.2904e-03
0.0005	TF4SPC	9.2288e-07
	4SPC	4.9834e-06
	TF3SPC	2.4643e-05
	EmbDP4(5)	1.0872e-04
0.00025	TF4SPC	2.2721e-08
	4SPC	4.9834e-07
	TF3SPC	9.7376e-06
	EmbDP4(5)	1.7494e-05

Table 2: Comparison between the proposed method and the other current methods for Problem 2.

STEP SIZE(H)	METHOD	MAX ERRORS
0.025	TF4SPC	2.2285e-05
	4SPC	3.7653e-04
	TF3SPC	3.0605e-03
	EmbDP4(5)	3.3496e-02
0.0125	TF4SPC	5.9309e-06
	4SPC	4.8701e-05
	TF3SPC	8.9456e-04
	EmbDP4(5)	9.6163e-03
0.01	TF4SPC	3.7987e-06
	4SPC	6.1242e-05
	TF3SPC	5.2224e-04
	EmbDP4(5)	4.2904e-03
0.005	TF4SPC	9.4943e-07
	4SPC	4.7416e-06
	TF3SPC	2.4643e-05
	EmbDP4(5)	1.0872e-04
0.0025	TF4SPC	2.3766e-08
	4SPC	2.3042e-07
	TF3SPC	9.7376e-06
	EmbDP4(5)	1.7494e-05

Table 3: Comparison between the proposed method and the other current methods for Problem 3.

STEP SIZE(H)	METHOD	MAX ERRORS
0.01	TF4SPC	3.2770e-04
	4SPC	1.7022e-03
	TF3SPC	6.0999e-03
	EmbDP4(5)	3.0549e-02
0.005	TF4SPC	2.3160e-06
	4SPC	4.9552e-05
	TF3SPC	1.5990e-04
	EmbDP4(5)	1.5406e-03
0.025	TF4SPC	1.2495e-07
	4SPC	4.9966e-06
	TF3SPC	1.3512e-05
	EmbDP4(5)	2.3893e-04
0.0125	TF4SPC	3.2643e-08
	4SPC	3.7242e-07
	TF3SPC	8.2423e-06
	EmbDP4(5)	1.4894e-05
0.001	TF4SPC	2.0908e-09
	4SPC	3.0002e-08
	TF3SPC	1.3284e-07
	EmbDP4(5)	2.3824e-06

We expect that the number of function evaluations for TF4SPC and 4SPC are the same because they are four-step methods, TF3SPC will have a lower number of function evaluations since it is a three-step method. Our main focus here is the accuracy of the methods, so we do not tabulate the number of function evaluations in tables 1-3.

5 Conclusions

From tables 1-3, we observed that the accuracy for the new proposed method is higher by one order for all the problems and for all the step size chosen compared to the original non-fitted method. The new proposed scheme has accuracy higher by order two compared to the TF3SPC method, though the TF3SPC is also a fitted method but it is of algebraic order four. The new scheme is more accurate compared to the EmbDP4(5) though the EmbDP4(5) method is of the same algebraic order (order five) as the original 4SPC and the new TF4SPC methods.

Considering the interval of integration is very large, that is [0,1000], we can conclude that trigonometric-fitting the 4-step predictor corrector method do improved the accuracy of the solution and hence the method is suitable for solving oscillatory problems.

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Conflicts of Interest The authors declare that no conflict of interests occurs.

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